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On the rectilinear motion of an inextensible string

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SUMMARY

In this paper we investigate the rectilinear motion of a string or chain with no bending stiffness, which is arranged in a straight line and bent double. We focus our attention on phenomena which are virtually possible at the kink and examine two mechanisms, one of which is dissipative and the other is not. The results of our calculations are compared with analogous computations in the literature.

1. Introduction

Once a problem in mechanics has been solved, usually one tries to interpret the results from a physical point of view. In general findings conform more or less to what one expects. However, there are exceptions in which the analytical or numerical inferences are unexpected in a sense. In what follows we refer to two examples where such results arise from the application of an improper model to a problem.

In his book, Rosenberg [1] explains the crack of the whip. He treats the latter as an inextensible uniform string with no bending stiffness which is arranged in a straight line and bent double as shown in Fig. 1. The distance of one end (subsequently we will call it the upper end) from some fixed datum is y and that of the other end (the lower end) is x. He assumes

$$x = x_0 - vt, \tag{1.1}$$

where v is a positive constant. The lower end thus moves uniformly to the right with a constant velocity.

According to [1] under these circumstances the velocity $\dot{y}$ of the upper end increases beyond

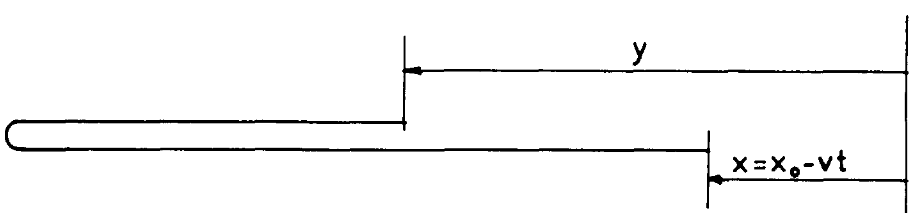

Figure 1. An inextensible uniform string arranged in a straight line and bent double. The lower part moves uniformly to the right with a constant velocity v.

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all bounds, even if its initial velocity is zero. It would tend to infinity when the string stretches out completely. Thus a motion of the lower end to the right would induce a motion of the upper part as a whole to the left. Noting the inability of a string or chain to sustain a pressure force over a finite distance, an assumption usually made, we are tempted to believe that particles of the upper end of the string will stay at rest until they are transferred to the lower part at the kink. The same problem has been treated by Kucharski [2], who arrives at the same conclusion as Rosenberg. Moreover he presents a collection of solutions of similar problems again with findings which are rather curious, e.g. a tensile force applied to one end of a string lying at rest, automatically provokes a tensile force in the other end.

In this paper we will consider two possible mechanisms which enable the apparent discontinuity of velocity of a particle passing through the kink. The first one is the most plausible mechanism, namely that of completely inelastic collisions between particles from the upper and the lower part. We will call this mechanism the dissipative model, and in the next section it is shown that this model accounts for what one expects to be the normal behaviour of a string or chain under various loading conditions. At any case the application of this model, which was not considered in [1] or in [2], removes the anomalies referred to above. The second model is non-dissipative and should be implemented by fitting a small circular disc at the kink. For a very small, but finite value of the radius r of the disc this allows for rapid, but continuous changes of the velocities of particles passing over it. This model, being somewhat artificial yet legitimate, has been considered in [2] as well. However, the conclusions arrived at in [2] do not agree with those we draw in Section 4.

To conclude with we note that the authors of [3] treat similar problems, e.g. that of a chain falling from a pile of links. In fact they use the dissipative model, however, as they apply directly the principle of linear momentum to the entire system, they need not consider the dissipative force explicitly.

2. The dissipative model

2.1 Calculation of the dissipative forces

Instead of considering the rheonomic problem of [1], we turn to a version somewhat more general. We suppose that the lower end of the string is loaded by a prescribed force $P(t) \geqslant 0$ and the upper end by a force $Q(t) \geqslant 0$, where t denotes time (Fig. 2). As before the string is

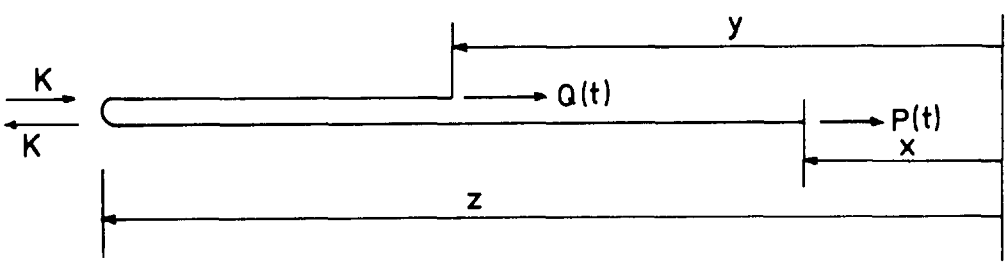

Figure 2. An inextensible uniform string arranged in a straight line and bent double. The external forces are $P(t)$ and $Q(t)$.

assumed to be inextensible and to be unable to transmit pressure forces over a finite distance. Evidently this problem is scleronomic and has two degrees of freedom in the Lagrangian formalism. As generalized coordinates we take the coordinates x and y depicted in Fig. 2. The distance z of the kink from the fixed datum is

$$z = \tfrac{1}{2}\ (\ell + x + y), \tag{2.1}$$

where ℓ is the constant length of the string. The length of the lower end is

$$z - x = \tfrac{1}{2}\ (\ell - x + y), \tag{2.2}$$

and that of the upper end is

$$z - y = \tfrac{1}{2}\ (\ell + x - y). \tag{2.3}$$

We assume that the two ends exert normal forces K on each other as indicated in Fig. 2. In the first instance the magnitude of K is unknown. In order to calculate it we have to distinguish two cases, viz.

$$\dot{y} - \dot{x} \geqslant 0, \tag{2.4}$$

and

$$\dot{y} - \dot{x} \leqslant 0, \tag{2.5}$$

where $\dot{x}$ and $\dot{y}$ denote the velocities of the two parts. The two cases are mutually exclusive unless $\dot{x} = \dot{y}$.

We first assume (2.4). According to (2.2) and (2.3) this means that the length of the lower end increases and that the upper part becomes shorter. In an interval of time Δt a quantity of mass $\frac{1}{2}\ \mu(\dot{y} - \dot{x})\Delta t$, where μ is the mass of the string per unit of length, is transferred from above to the lower part. Instantaneously its velocity changes from $\dot{y}$ to $\dot{x}$, so that the increase of momentum of that mass is $-\frac{1}{2}\ \mu(\dot{y} - \dot{x})^2\Delta t$. This change is brought about by the force K exerted by the lower part on particles leaving the upper end. In view of the fact that this pressure force cannot be transferred by the upper part over a finite distance, it is absorbed locally by the quantity of mass which discontinuously changes its velocity. Equating the increase of momentum and the impulse $-K\Delta t$ yields

$$K = +\tfrac{1}{2}\mu(\dot{y} - \dot{x})^2. \tag{2.6}$$

We note that the force K exerted on the lower end has to be considered as being the boundary value at z of a normal tensile force distributed continuously throughout the lower part.

It remains to prove that the two forces K together dissipate energy if $\dot{y} - \dot{x} > 0$. For that purpose we apply well-known formulae for the work dA done by the forces K in an interval of time dt

$$dA = K\ dt.\ \tfrac{1}{2}(-\dot{y} - \dot{x}) + K.\dot{x}dt = -\tfrac{1}{2}K(\dot{y} - \dot{x})dt, \tag{2.7}$$

The first term of the second expression represents the work done by the force K on particles leaving the upper end, and the second one refers to the work done by the lower K. The single terms have no fixed sign, however, their sum is always negative as $\dot{y}-\dot{x}>0$. Combining (2.6) and (2.7) we have

$$dA = -\tfrac{1}{4}\mu(\dot{y}-\dot{x})^3\,dt. \tag{2.8}$$

Hence, in general there is dissipation of energy at the kink.

The value of K in the case (2.5) follows from the preceding results. To this end we interchange the symbols x and y, obtaining again the condition (2.4), and observe the change of sign of the forces K. In this way we find

$$\dot{y}-\dot{x}\leqslant 0, \quad K = -\tfrac{1}{2}\mu(\dot{y}-\dot{x})^2, \quad dA = +\tfrac{1}{4}\mu(\dot{y}-\dot{x})^3\,dt. \tag{2.9}$$

2.2 Calculation of the generalized forces

It is evident that the forces K are to be considered as impressed forces which contribute to the virtual work $\delta\mathscr{A}$ as a result of any virtual displacements δx and δy assigned to the system. According to the expression (2.7) for the real work dA, the expression for the virtual work $\delta\mathscr{A}$ must have the following form

$$\delta\mathscr{A} = -\tfrac{1}{2}K\quad\delta y + \tfrac{1}{2}K\,\delta x. \tag{2.10}$$

Hence, the generalized forces Q_x and Q_y resulting from K are

$$Q_x = +\tfrac{1}{2}K \quad\text{and}\quad Q_y = -\tfrac{1}{2}K. \tag{2.11}$$

This is true irrespective of the sign of the relative velocity $\dot{y}-\dot{x}$. In this way we find using (2.6) and (2.9)[2]

$$\dot{y}-\dot{x}\geqslant 0, \quad Q_x = +\tfrac{1}{4}\mu(\dot{y}-\dot{x})^2, \quad Q_y = -\tfrac{1}{4}\mu(\dot{y}-\dot{x})^2, \tag{2.12}$$

$$\dot{y}-\dot{x}\leqslant 0, \quad Q_x = -\tfrac{1}{4}\mu(\dot{y}-\dot{x})^2, \quad Q_y = +\tfrac{1}{4}\mu(\dot{y}-\dot{x})^2. \tag{2.13}$$

2.3 The Lagrangian formalism

We apply the Lagrangian formalism to the problem of Section 2.1 (Fig. 2). The following expressions for the kinetic energy T and the potential energy U are used

$$T = \tfrac{1}{4}\mu\,[(\ell - x + y)\,\dot{x}^2 + (\ell + x - y)\,\dot{y}^2], \tag{2.14}$$

$$U = P(t)x + Q(t)y. \tag{2.15}$$

If $\dot{y} - \dot{x} \geqslant 0$, then through the use of (2.12) the Lagrangian equations become

$$\begin{aligned}
&\frac{d}{dt}\left[\tfrac{1}{2}\mu(\ell + x - y)\dot{y}\right] - \tfrac{1}{4}\mu(\dot{x}^2 - \dot{y}^2) = -\tfrac{1}{4}\mu(\dot{y} - \dot{x})^2 - Q(t),\\
&\frac{d}{dt}\left[\tfrac{1}{2}\mu(\ell - x + y)\dot{x}\right] + \tfrac{1}{4}\mu(\dot{x}^2 - \dot{y}^2) = +\tfrac{1}{4}\mu(\dot{y} - \dot{x})^2 - P(t).
\end{aligned} \tag{2.16}$$

Adding both sides of these equations we obtain the balance of linear momentum of the entire system. From (2.16) we find

$$\begin{aligned}
&\tfrac{1}{2}\mu(\ell + x - y)\ddot{y} = -Q(t),\\
&\tfrac{1}{2}\mu(\ell - x + y)\ddot{x} = +\tfrac{1}{2}\mu(\dot{y} - \dot{x})^2 - P(t).
\end{aligned} \tag{2.17}$$

The first equation shows that the normal force in the upper end at a cross-section just to the right of the kink is indeed zero. The analogous force in the lower part is a tensile force $\frac{1}{2}\mu(\dot{y} - \dot{x})^2$, as was to be expected. These findings differ from those in [2]. If $\ddot{x} = 0$ and $\ddot{y} = 0$, then we have

$$P(t) = +\tfrac{1}{2}\mu(\dot{y} - \dot{x})^2 \quad \text{and} \quad Q(t) = 0. \tag{2.18}$$

As to the whip problem of Rosenberg, from $(2.17)^1$ we find for $Q(t) = 0$ the result $\ddot{y} = 0$, as long as the length of the upper end $\ell + x - y > 0$. Evidently the motion of the upper end is uniform in time (whatever the motion of lower part may be), and there are no velocities increasing beyond all bounds.

2.4 A simple motion

Finally we apply (2.17) to the following problem (Fig. 3). The motion sketched in Fig. 3 developes if we apply a constant force $P > 0$ to the lower end, starting from a state of rest in which the string is stretched out completely. We omit trivial calculations and only mention the following results:

$$\begin{aligned}
&0 \leqslant t \leqslant \ell\sqrt{\frac{2\mu}{P}}, \quad x = \ell - t\sqrt{\frac{2P}{\mu}}, \quad \dot{x} = -\sqrt{\frac{2P}{\mu}}, \quad K = P, \ Q = 0,\\
&T = \tfrac{1}{2}P\sqrt{\frac{2P}{\mu}}\,t, \quad U = -P\sqrt{\frac{2P}{\mu}}\,t, \quad \text{so that } \dot{T} + \dot{U} = -\tfrac{1}{2}P\sqrt{\frac{2P}{\mu}}.
\end{aligned} \tag{2.19}$$

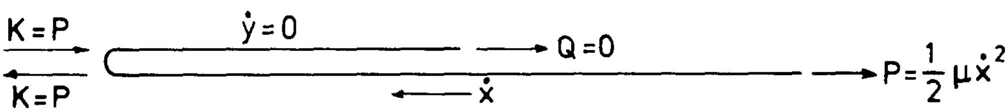

Figure 3. A simple motion of the string possible if the dissipative model is applied.

The latter result agrees with the rate at which energy is dissipated by the forces K according to (2.8)

$$dA = -\tfrac{1}{2} P \sqrt{\frac{2P}{\mu}}\, dt. \tag{2.20}$$

In passing we note that in a frame of reference moving with a velocity $(2P/\mu)^{\frac{1}{2}}$ to the right, which is also Galilean, the motion is transformed to that of a whip with $v = 0$ and the upper end moving to the left.

The case $\dot{y} - \dot{x} \leqslant 0$ does not yield further insight into the behaviour of the dissipative model and hence can be left out of consideration.

3. The non-dissipative model

3.1 The Lagrangian formalism

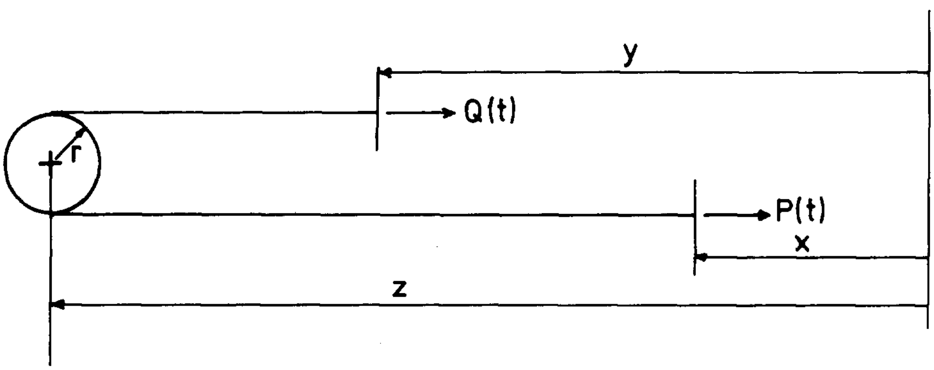

Figure 4. An inextensible uniform string passing over a circular disc. The two ends and the centre of the disc execute a rectilinear motion as a result of the applied forces $P(t)$ and $Q(t)$.

First we review the non-dissipative model proposed by Kucharski [2]. The usual assumptions on the motion of the string, viz. it is rectilinear and uniform, are retained. However, the conditions at the kink are different from those in the preceding chapter. At this point the system is assumed to be provided with a small circular disc over which the string passes. The disc can move in the same direction as the ends of the string, and in addition it rotates so that there is no slip between the disc and the string. In this way the local discontinuity of the velocity at the kink has been replaced by a rapid, but gradual change, at least as long as the radius r of the disc is finite. On this condition there is no dissipation. As before it is understood that the string cannot stand a pressure force. The limitations of this model follow readily from applying the Lagrangian equations to a simple motion.

To this end we use the following expressions for the kinetic energy T and the potential U

$$\begin{aligned} T &= \tfrac{1}{4}\mu\,[(\ell - \pi r - x + y)\dot{x}^2 + (\ell - \pi r + x - y)\dot{y}^2 + \pi r(\dot{x}^2 + \dot{y}^2)], \\ U &= P(t)x + Q(t)y, \end{aligned} \tag{3.1}$$

where we have neglected the mass of the disc. We note that there is no need to discriminate here between the conditions (2.4) and (2.5). The Lagrangian equations are found to be

$$\begin{aligned}&\frac{d}{dt}\left[\tfrac{1}{2}\mu(\ell-\pi r-x+y)\dot{x}+\tfrac{1}{2}\mu\pi r\dot{x}\right]+\tfrac{1}{4}\mu(\dot{x}^2-\dot{y}^2)=-P(t),\\&\frac{d}{dt}\left[\tfrac{1}{2}\mu(\ell-\pi r+x-y)\dot{y}+\tfrac{1}{2}\mu\pi r\dot{y}\right]-\tfrac{1}{4}\mu(\dot{x}^2-\dot{y}^2)=-Q(t),\end{aligned}\tag{3.2}$$

from which the balance of linear momentum follows by adding both sides of the equations. Reducing (3.2) we obtain

$$\begin{aligned}&\tfrac{1}{2}\mu(\ell-x+y)\ddot{x}=+\tfrac{1}{4}\mu(\dot{x}-\dot{y})^2-P(t),\\&\tfrac{1}{2}\mu(\ell+x-y)\ddot{y}=+\tfrac{1}{4}\mu(\dot{x}-\dot{y})^2-Q(t).\end{aligned}\tag{3.3}$$

It appears that the normal force in a cross-section of the lower end just to the right of the disc is

$$\tfrac{1}{4}\mu(\dot{x}+\dot{y})^2-\tfrac{1}{2}\mu r\ddot{x},\tag{3.4}$$

and that in the upper part

$$\tfrac{1}{4}\mu(\dot{x}-\dot{y})^2-\tfrac{1}{2}\mu r\ddot{y}.\tag{3.5}$$

If we impose the condition that these forces must be non negative, then we find certain bounds for the maximal accelerations which can be allowed. For vanishingly small values of r these results are to be compared with the analogous findings from (2.17). If $\ddot{x}=0$ and $\ddot{y}=0$, then

$$P(t)=\tfrac{1}{4}\mu(\dot{x}-\dot{y})^2\quad\text{and}\quad Q(t)=\tfrac{1}{4}\mu(\dot{x}-\dot{y})^2,\tag{3.6}$$

and we observe the difference from (2.18).

3.2· A simple motion

We again consider the problem of Section 2.4 (Fig. 5).

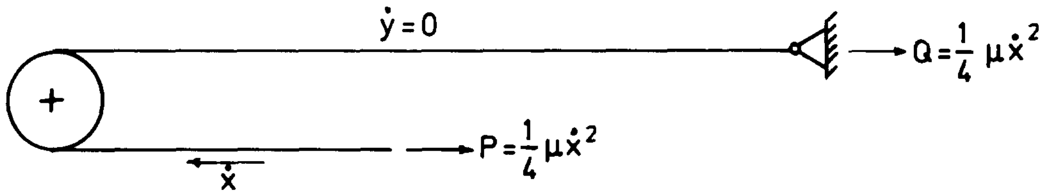

Figure 5. A simple motion of the string possible if the non dissipative model is applied.

Suppressing trivial calculations we give the following results, valid for $r \to 0$,

$$0 \leqslant t \leqslant \ell\sqrt{\frac{\mu}{P}}, \quad x = \ell - 2t\sqrt{\frac{P}{\mu}}, \quad \dot{x} = -2\sqrt{\frac{P}{\mu}}, \quad Q = P(=\tfrac{1}{4}\mu\dot{x}^2),$$
$$T = 2P\sqrt{\frac{P}{\mu}}\,t, \quad U = -2P\sqrt{\frac{P}{\mu}}\,t, \tag{3.7}$$

so that $\dot{T} + \dot{U} = 0$. Apparently there is no dissipation.

4. Discussion and conclusions

It follows from the preceding calculations that discriminating between the two possible models is by no means superfluous. According to Sections 2.4 and 3.2, the models behave differently under identical loading conditions and give rise to varying motions of the system. If we ask which of the two models simulates the real behaviour of the system best, we first have to settle whether either of the two can be applied in a certain class of problems at all. To amplify this, we compare (2.19) with (3.7), which both pertain to the simple motion during which $\dot{y} = 0$ and the force P applied to the lower end is constant. As has been noted in Section 2.4, this motion is equivalent to that occurring in the whip problem. Observing that the use of the non-dissipative model implies $Q(t) > 0$, we note that this condition is not complied with in [1] and [2] where the whip problem is considered. Hence, the related results contained in these references are not relevant to this problem. However, if the right-hand end of the upper part is a fixed point, so that a tensile force Q can indeed develop, we are entitled to use either of the two models in the calculation of the problems of Section 2.4 and 3.2: neither is excluded beforehand. If we then compare (2.19) with (3.7) we see that, in order to impart one and the same velocity to the lower end, the driving force P must be twice as large in the case of the dissipative model as it is in the non-dissipative one. This result sounds plausible. For the rest it seems that the potential applicability of the non-dissipative model is less than that of the dissipative one. By its very properties the latter mechanism is automatically compatible with the uniform motion of the string. To enlarge the potential application of the non-dissipative model, we should for example no longer insist upon the uniform motion of the string, but allow for longitudinal vibrations. In the case of the whip, however, it seems that we cannot dispense with the flexural motion of the string. Rotary inertia has probably to be considered as well.

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